Non perturbative renormalization group potentials and quintessence

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New solutions to the non preturbative renormalization group equation for the effective action of a scalar field theory in the Local Potential Approximation having the exponential form $e^{\pm\phi}$ are found. This result could be relevant for those quintessence phenomenological models where this kind of potentials are already used, giving them a solid field theoretical derivation. Other non perturbative solutions, that could also be considered for the quintessence scenario, are also found. Apart from this particular cosmological application, these results could be relevant for other models where scalar fields are involved, in particular for the scalar sector of the standard model.

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One of the most challenging problems in fundamental physics has been the search for a theoretical argument, hopefully a symmetry, that could explain the vanishing of the cosmological constant [1]. Recent observations from high redshift supernovae [2], combined with the data on the fluctuation of the cosmic microwave background [3], have changed our perspective. Apparently Ω_M, the ratio of the barionic and cold dark matter density to the critical density, is about $\frac{1}{3}$. This means that either the universe is open or the missing energy is provided by some new form of matter. The simplest candidate is a cosmological constant term. Alternative models where the missing energy is given by a scalar field slowly rolling down its potential, the "quintessence" [4], have recently attracted lot of attention.

For a truly constant vacuum energy term the "old" problem of explaining the vanishing of the cosmological constant is replaced by the equally difficult one of explaining why it has the small observed value of about $(3\times 10^{-3}ev)^4$. In the quintessence scenario the equivalent problem is the so called "coincidence problem". The matter and the scalar fields evolve differently but we observe today an order of magnitude "coincidence" between the matter energy density Ω_M and the quintessence energy density Ω_ϕ that requires a fantastic fine tuning of the initial conditions. The notion of "tracker field" [5], a quintessence scalar field that evolves to an attractor solution during its rolling down, has been introduced to circumvent this problem. For a wide range of initial conditions the attractor is stable. An explanation for the coincidence can be obtained this way and this has been advocated as an argument in favour of the quintessence scenario [5]. Recently another interesting argument has been given in the framework of the brane-world picture [6].

Even restricting ourselves to tracker fields there is still a large amount of arbitrariness in the possible form of this potential. Different proposals [4,5,7–12] have been essentially based on their capability to reproduce the observational data. Also interesting attempts have been done [13–17] to derive its form from particle physics models, but they still leave the door open to several possibilities and it is not possible to discriminate between the different proposals.

In this Letter I show that the renormalization group equation for the effective action of a single component scalar field theory in the Local Potential Approximation (LPA) possesses non perturbative solutions in addition to the well known perturbative one. These are of the form of exponential potentials that are among the favoured candidates in the quintessence scenario [4–6,9,11]. I think that this result can give a strong motivation to these potentials. I will comment more on this point later.

The exact renormalization group equation for the effective action has been found in Ref. [18]. By considering only the potential term, an approximation to this equation (the LPA) is obtained [19], and for a single component scalar field theory in $d=4$ dimensions it reads:

$$k\frac{\partial}{\partial k}U_k(\phi) = -\frac{k^4}{16\pi^2}ln\Big(\frac{k^2+U_k^{''}(\phi)}{k^2+m^2}\Big). \tag{1}$$

Here k is the current scale, m^2 is a constant with dimension $(\text{mass})^2$, U_k is the potential at the scale k and the $'$ means derivative with respect to the field ϕ.
Eq.(1) is a non perturbative evolution equation for U_k. It is immediate to see that the k-independent solution to Eq.(1), i.e. the fixed point potential, is the trivial gaussian one:

$$U_f(\phi) = \frac{1}{2}m^2\phi^2 + \alpha\phi + \beta. \tag{2}$$

One simple way to show that Eq.(1) admits the well known perturbative solution is as follows. Let's consider first a small deviation of the potential $U_k(\phi)$ around $U_f(\phi)$,

$$U_k(\phi) = U_f(\phi) + \delta U_k(\phi). \tag{3}$$

We develop now the logarithm in Eq.(1) in powers of δU_k and expand δU_k in powers of the field (for the sake of simplicity we consider a potential with the $Z(2)$ symmetry $\phi\to-\phi$)

$$\delta U_k(\phi) = \frac{\lambda_2(k)}{2}\phi^2 + \frac{\lambda_4(k)}{4!}\phi^4 + \cdots. \tag{4}$$

At any order in δU_k we obtain this way an infinite system of equations for the coupling constants. By truncating this system to the first two equations for $\lambda_2(k)$ and $\lambda_4(k)$ and solving iteratively up to the second order in δU_k [20] we get the known perturbative one loop RG flow for the coupling constants. Extrapolating down to $k = 0$, this flow identifies the gaussian fixed point as infrared stable, i.e. we have the standard result about the triviality of the theory.

We ask now the question about the possibility of having non perturbative solutions. The only k-independent solution to Eq.(1), i.e. the only fixed point potential, is the gaussian one found in Eq.(2).

We want to linearize now Eq.(1) around U_f and look for a *small* but *non perturbative* δU_k. We have (for notational simplicity I write U_k rather than δU_k):

$$k\frac{\partial}{\partial k}U_k(\phi) = -\frac{k^4}{16\pi^2}\frac{1}{k^2+m^2}U_k^{''}(\phi). \tag{5}$$

There is a class of non perturbative solutions to Eq.(5) that is very easy to find. Let's seek for solutions of the form:

$$U_k(\phi) = f(k)g(\phi). \tag{6}$$

Once inserted in Eq.(5), the ansatz (6) gives:

$$\frac{d^2g(\phi)}{d\phi^2} - \frac{1}{\mu^2}g(\phi) = 0 \tag{7}$$

$$\frac{df(k)}{dk} + \frac{1}{16\pi^2\mu^2}\frac{k^3}{k^2+m^2}f(k) = 0, \tag{8}$$

where μ^2 is any constant with dimension (mass)^2 that allows to separate Eq.(5) in the two Eqs. (7) and (8).

Solving now these equations is a simple exercise. For positive values of the constant μ^2 the solutions to Eq.(5) have the form

$$U_k(\phi) = M_1^4\ e^{-\frac{1}{32\pi^2\mu_1^2}\left(k^2-\ m^2\ln\frac{k^2+m^2}{m^2}\right)}e^{-\frac{\phi}{\mu_1}} \tag{9}$$

$$U_k(\phi) = M_2^4\ e^{-\frac{1}{32\pi^2\mu_2^2}\left(k^2-\ m^2 ln\frac{k^2+m^2}{m^2}\right)}e^{+\frac{\phi}{\mu_2}}, \tag{10}$$

for arbitrary values of the mass dimension constants M_i and μ_i.

For negative values of μ^2 (calling $\bar{\mu}^2 = -\mu^2 > 0$), the solutions to Eq.(5) have the form:

$$U_k(\phi) = M_3^4\ e^{+\frac{1}{32\pi^2\bar{\mu}_3^2}\left(k^2-\ m^2 ln\frac{k^2+m^2}{m^2}\right)}\cos\Big(\frac{\phi}{\bar{\mu}_3}\Big) \tag{11}$$

$$U_k(\phi) = M_4^4\ e^{+\frac{1}{32\pi^2\bar{\mu}_4^2}\left(k^2-\ m^2 ln\frac{k^2+m^2}{m^2}\right)}\sin\Big(\frac{\phi}{\bar{\mu}_4}\Big). \tag{12}$$

For solutions of the kind (9) and (10) the gaussian potential $U_f(\phi)$ is an ultraviolet fixed point. From that we immediately understand the completely different nature of these solutions with respect to the perturbative one.

As already mentioned these exponential potentials $e^{\pm\phi}$, as well as linear combinations of them, are among the favourite candidates for the quintessence scenario [4,5,11].

The attempts that have been done to derive different forms of quintessence potentials all started from some sort of "fundamental" higher energy model. For example, inverse power-law potentials have been motivated from Supersymmetric QCD [13,14]. For another derivation of the same kind of potentials see [15]. Exponential potentials of the form found above arise naturally in several higher energy/higher dimensional theories [21]. As we simply don't know the theory that describes our world at very high energies, the use of phenomenologically motivated potentials is certainly well justified. In that respect indications coming directly from the effective theory of the quintessence field should be considered as very welcome. Actually, whatever the structure of the higher energy/higher dimensional theory, i.e. whatever the fundamental origin of the scalar quintessence field is, the "low energy" effective theory for this field should be very well described by Eq.(1). This happens because the higher energy degrees of freedom decouple from the quintessence field. There could still be the problem of the coupling of this field to ordinary matter, i.e. to ordinary standard model fields. As the long range forces that these couplings would generate are not observed, we can suppose that they are suppressed through some mechanism as for instance the one proposed in Ref. [22]. In this case the renormalization group Eq.(1) gives the flow equation for the quintessence field irrespectively of the nature of the higher energy theory and the above results (9) and (10) give a solid motivation from the "low energy side" to the phenomenological exponential potentials $e^{\pm\phi}$.

Concerning the solutions of the kind (11) and (12), we see that the gaussian potential $U_f(\phi)$ is neither an infrared nor an ultraviolet fixed point for them, even though we can multiply these solutions times a small dimensionless constant so that they still make sense as linearized solutions of the Eq.(1). I want to mention here that cosine potentials have also been considered as possible quintessence candidates [7,4].

We want to seek now for other non perturbative solutions. Actually there is at least another class of such solutions that can be easily found. To see that, we switch first to the dimensionless form of Eq.(1). If we define the dimensionless field φ, the dimensionless scale parameter t and the dimensionless potential $v(\varphi,t)$ from

$$\varphi = \frac{1}{4\pi}\frac{\phi}{k}, \qquad t = \ln\frac{\Lambda}{k}, \qquad U_k(\phi) = \frac{k^4}{16\pi^2}v(\varphi,t), \tag{13}$$

where Λ is a boundary value for k, Eq.(1) becomes:

$$\frac{\partial v}{\partial t} + \varphi\frac{\partial v}{\partial\varphi} - 4v = ln\left(\frac{1+\frac{\partial^2 v}{\partial\varphi^2}}{1+\frac{m^2}{\Lambda^2}e^{2t}}\right). \tag{14}$$

The dimensionless potential $v_f(\varphi, t)$ that corresponds to the gaussian potential $U_f(\phi)$ of Eq.(2) is:

$$v_f(\varphi, t) = \frac{1}{2}\frac{m^2}{\Lambda^2}e^{2t}\varphi^2 + \frac{4\pi\alpha}{\Lambda^3}e^{3t}\varphi + \frac{16\pi^2\beta}{\Lambda^4}e^{4t}, \quad (15)$$

and solves the equation:

$$\frac{\partial v}{\partial t} + \varphi\frac{\partial v}{\partial \varphi} - 4v = 0. \quad (16)$$

Actually Eq.(14) is more often written as [23]:

$$\frac{\partial v}{\partial t} + \varphi\frac{\partial v}{\partial \varphi} - 4v = ln\left(1 + \frac{\partial^2 v}{\partial \varphi^2}\right), \quad (17)$$

i.e. by setting $m^2 = 0$. This simply corresponds to choose a massless fixed point potential and from now on we also restrict ourselves to this case. We consider now a small fluctuation around the fixed point potential,

$$v(\varphi, t) = v_f(\varphi, t) + \delta v(\varphi, t), \quad (18)$$

and linearize Eq.(17) around v_f (again we write $v(\varphi, t)$ rather than $\delta v(\varphi, t)$) to get:

$$\frac{\partial v}{\partial t} + \varphi\frac{\partial v}{\partial \varphi} - 4v = \frac{\partial^2 v}{\partial \varphi^2}. \quad (19)$$

By following the same strategy as before we look for solutions to Eq.(19) with factorized t and φ dependence:

$$v(\varphi, t) = f(t)g(\varphi). \quad (20)$$

Inserting the ansatz (20) in Eq.(19) we have:

$$\frac{d^2 g(\varphi)}{d\varphi^2} - \varphi\frac{dg(\varphi)}{d\varphi} + 4g(\varphi) = \alpha g(\varphi) \quad (21)$$

$$\frac{df(t)}{dt} = \alpha f(t), \quad (22)$$

where α is an arbitrary dimensionless constant that allows to separate Eq.(19) in the two Eqs. (21) and (22). Equation (22) is trivially solved and gives:

$$f(t) = Ae^{\alpha t}, \quad (23)$$

where A is the integration constant.
The solution to Eq.(21) can be found by series. Writing

$$g(\varphi) = \sum_{n=0}^{\infty} c_n \varphi^n, \quad (24)$$

inserting (24) in Eq.(21) and exploiting the recurrence relations between the coefficients c_n, we get the two linearly independent solutions ($a = \alpha - 4$):

$$g_1(\varphi) = c_0\left(1 + \sum_{n=1}^{\infty}\frac{\Pi_{j=1}^{n}(a + 2j - 2)}{(2n)!}\varphi^{2n}\right) \quad (25)$$

$$g_2(\varphi) = c_1\left(\varphi + \sum_{n=1}^{\infty}\frac{\Pi_{j=1}^{n}(a + 2j - 1)}{(2n+1)!}\varphi^{2n+1}\right) \quad (26)$$

where I have explicitly kept c_0 and c_1, the first two coefficients in Eq.(24), as integration constants.

After some trivial algebra we see that Eqs.(25) and (26) can be written in terms of confluent hypergeometric functions and the general solution to Eq.(21) takes the compact form

$$g(\varphi) = c_0 M\left(\frac{a}{2}, \frac{1}{2}, \frac{\varphi^2}{2}\right) + c_1\varphi M\left(\frac{a+1}{2}, \frac{3}{2}, \frac{\varphi^2}{2}\right). \quad (27)$$

We recall here that the confluent hypergeometric function $M(a, b, x)$ is defined as

$$M(a, b, x) = 1 + \frac{a}{b}x + \frac{1}{2!}\frac{a(a+1)}{b(b+1)}x^2 + \cdots. \quad (28)$$

Of course once dimensionless solutions to Eq.(19) are known, it is a trivial exercise to reconstruct dimensionfull potentials from Eq.(13). Some of the potentials obtained this way could also be considered as quintessence candidates.

It is worth to compare the general solution (27) to Eq.(21) with a non perturbative result that has been obtained some years ago by Halpern and Huang [24,25] following a different but essentially equivalent approach. Searching for alternatives to the trivial ϕ^4 theory, the authors expanded the potential in even powers of the field and derived an infinite system of differential equations for the coupling constants. Looking for new eigendirections in the parameter space, they actually ended with the solution $g_1(\varphi)$ to Eq.(21). More precisely, after making the trivial changes to match the two different notations and restricting ourselves to consider the $N = 1$ and d=4 case as in the present paper, we can immediately check that eq.(49) of Ref. [25] coincides with the $g_1(\varphi)$ solution above. As these authors considered potentials containing only even powers of the field, obviously they could only get the solution $g_1(\varphi)$.

At a first sight it could seem that the solution $g_2(\varphi)$ should be discarded as it contains odd powers of the field and as such it is unbounded from below. We would conclude in this case that the only physically acceptable general solution to Eq.(19) is $g_1(\varphi)$, i.e. the Halpern-Huang result. But this is not always true. It is immediate to see from Eq.(27) that for all the positive or negative integer odd values of α such that $\alpha < 4$, $g_2(\varphi)$ is a polynomial in φ and it can be combined with $g_1(\varphi)$ to give a bounded from below potential. It is also easy to give examples where even when the hypergeometric function in $g_2(\varphi)$ keeps all the infinite terms, still a linear combination of $g_1(\varphi)$ and $g_2(\varphi)$ gives a bounded from below potential. Take for instance the case $\alpha = 5$ for which both g_1 and g_2 are not polynomials. From the asymptotic behaviour of the hypergeometric function we easily see that c_0 and

c_1 can be chosen in such a way that the resulting potential is bounded from below. The class of physically acceptable potentials, that are solutions of the Eq.(21), is larger than that spanned by g_1 only.

I should mention at this point that an attempt to solve Eq.(19) has been recently made in Ref. [26] where the solution $v(\varphi, t) = e^{5t} e^{\frac{\varphi^2}{2}}$ is presented [27]. The author says that this solution is asymptotically similar to those of Ref. [24] but that the connection between the two results is unclear to him. From Refs. [24,25] and from Eq.(27) above (setting $c_1 = 0$) we immediately see that this solution is just the Halpern-Huang result for $\alpha = 5$. It is also a trivial exercise to verify that the other solutions presented in [26] (see Eq.(21) of that paper) are particular cases of the general solution, Eq.(27), obtained for the integer values $\alpha = 6, 7, 8, \cdots$.

To summarize in this letter I have presented new solutions to the LPA of the exact renormalization group equation for the effective action of a single component scalar field theory. These potentials have the exponential form $e^{\pm\phi}$ and have been recently used in phenomenological quintessence models. As the effective theory for the quintessence field should be governed by Eq.(1) whatever its higher energy origin, I argue that the fact that they arise as solutions to the renormalization group equation gives a *derivation* of these potentials that is alternative and complementary to those based on higher energy theories and should allow to discriminate between different proposals. I have also presented other solutions to the LPA renormalization group equation some of which already partially known [24,25].

Apart from the application to the particular cosmological problem suggested in the present paper, I think that these results could also be relevant in other frameworks where scalar fields play a role. For the non perturbative solutions to Eq.(19) of the kind $e^{\pm\phi}$, the gaussian potential is an UV stable fixed point. The same is true for those solutions of the kind (27) when $\alpha > 0$. This result is opposite to the perturbative one and its implications in particle physics models are certainly worth to explore. The scalar sector of the standard model is itself an open problem. It is not a priori clear whether the existence of these solutions could have some relevance for the theory. I hope to come back to this issue in a future paper. Work is in progress in this direction.

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